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Virtual Photon Structure to Nonleading Order in QCD

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ABSTRACT

We investigate the virtual photon structure functions for $\Lambda^2 \ll P^2 \ll Q^2$, where $-Q^2(-P^2)$ is the mass squared of the probe (target) photon. We do this to next-to-leading order in QCD. The nonleading corrections significantly modify the leading log result, in particular at large x. Also, the perturbatively calculated structure function is positive at low x even for $P^2 \approx 1$ GeV^2. (For a real photon target it becomes negative at low x.) For large P^2 the QCD result approaches the box diagram (Born) structure function also when nonleading contributions are included. It is, however, important to include the large nonleading box diagram contributions in making this comparison.

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I. INTRODUCTION

In the last few years, the real photon structure functions [1-2] have been much studied using quantum chromodynamics (QCD) [3-10]

Remarkably, the photon structure function is calculable in QCD--not merely the Q^2 dependence but also the shape and magnitude in the large Q^2 limit. This is due to the dominance of the term corresponding to a pointlike coupling of the photon over uncalculable terms arising from the hadronic component of the photon [2,3].

It is tempting to ask what happens to the structure function of a virtual photon with "mass" much larger than the QCD scale parameter Λ. Should we expect that then there are no uncalculable terms at all? In this paper we study this kinematical regime (Fig.1),

$$\Lambda^2 << P^2 << Q^2 \tag{1.1}$$

where $q^2=-Q^2<0$ ($p^2=-P^2<0$) is the mass squared of the probe (target) photon in the 2γ process, accessible in e^+e^- collisions [1]. The second inequality in (1.1) is to avoid the appearance of power corrections of the form $(P^2/Q^2)^k$ $(k=1,2,...)$.

In previous work [11], we studied virtual photon structure in the kinematic region (1.1) to leading log order in QCD. We summed up the leading logs, $(\alpha_s \ln Q^2/P^2)^n$, using the Altarelli-Parisi [12] type evolution equation.

In the present paper, we extend our previous calculation to nonleading order in QCD, this time using the operator product expansion (OPE) and renormalization group (RG) approach.* (The next-to-leading order calculation for a real photon target was done by Bardeen and Buras [7].)

In the kinematic region (1.1) the hadronic component on the photon can also be dealt with perturbatively. More precisely, we can apply the OPE to photon matrix elements of the hadronic operators, $\langle\gamma(p)|O_n^i|\gamma(p)\rangle$ for $i=\psi$ (quark singlet), NS (non-singlet). There remains no incalculable term, in contrast to the real photon case, where the unknown photon matrix element of the hadronic energy-momentum tensor appears [3,7].

Our main concern here is the virtual photon structure function. However, there is some relevance to the real photon case. If we introduce a normalization point for the hadronic operators and keep logarithmically decreasing terms, then we find that the Q^2 dependence is controlled by the same parameters, as well as by the unknown photon matrix element of the hadronic operators.

By inverting the moments we will see that in the limit (1.1) the nonleading corrections to the virtual photon structure functions are large -- especially at large x. It modifies the leading-log result significantly. The pertur-

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*See for example ref.[10].

batively calculated off-shell photon structure function is positive definite even at small x. The corresponding real photon structure function is negative there, as was shown in [8].

As already mentioned, we confine ourselves here to the limit (1.1) so as to be able to neglect power corrections of the form $(P^2/Q^2)^k$ (k=1,2,...) [13]. These arise from kinematical target mass effects [13,14] and also from higher-twist corrections. Experimentally, we expect more events for the region $P^2 \ll Q^2$ than for the doubly deep inelastic region $P^2 \sim Q^2$, so it is also of more practical interest. (At the end of this paper we will briefly comment on the $P^2 \sim Q^2$ region.)

The paper is organized as follows. In the next section we present the theoretical framework. In section 3, we calculate moments of the virtual photon structure functions including nonleading log QCD corrections. We discuss the real photon case in section 4. In section 5 we invert the moments for virtual and real photon structure functions and present the results. The last section is discussion.

II. THEORETICAL FRAMEWORK

Unless otherwise stated, we will follow the notation of Bardeen and Buras in our discussion [7].

Consider the forward virtual photon-photon scattering amplitude (Fig.2).

$$T_{\mu\nu\alpha\beta}(p,q) = i\int d^4x d^4y d^4z e^{iqx} e^{ip(y-z)}$$

$$\times \langle 0|T(J_\mu(x)J_\nu(0)J_\alpha(y)J_\beta(z))|0\rangle \; . \qquad (2.1)$$

The structure tensor is just the absorptive part of this amplitude,

$$W_{\mu\nu\alpha\beta}(p,q) = \frac{1}{\pi} \, \mathrm{Im} \, T_{\mu\nu\alpha\beta} =$$

$$= \frac{1}{2} \int d^4x d^4y d^4z \; e^{iqx} e^{ip(y-z)}$$

$$\times \langle 0|T^*(J_\mu(x)J_\alpha(y))T(J_\nu(0)J_\beta(z))|0\rangle \; . \qquad (2.2)$$

There are 8 independent structure functions in $W_{\mu\nu\alpha\beta}$, as discussed in refs.[18,19]. Here we will take a spin average for the target photon. The number of independent structure functions is thus reduced to two. Neglecting P^2/Q^2 corrections we call them F_2^γ and F_L^γ,

$$W^{\gamma}_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\lambda} \varepsilon^{\alpha *}_{(\lambda)}(p) W_{\mu\nu\alpha\beta}(p,q)\varepsilon^{\beta}_{(\lambda)}(p)$$

$$= \frac{1}{2} g^{\alpha\beta} W_{\mu\nu\alpha\beta}(p,q)$$

$$= \frac{1}{2}\int d^4x \; e^{iqx} \langle\gamma(p)|J_{\mu}(x)J_{\nu}(0)|\gamma(p)\rangle_{\text{spin av.}} \qquad (2.3)$$

and

$$W^{\gamma}_{\mu\nu}(p,q) = (g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}) \frac{1}{x} F^{\gamma}_{L}(x,Q^2,P^2)$$

$$+\left[p_{\mu}p_{\nu} - \frac{p\cdot q}{q^2}(p_{\mu}q_{\nu}+p_{\nu}q_{\mu}) + \frac{(p\cdot q)^2}{q^2} g_{\mu\nu}\right] \frac{2}{p\cdot q} F^{\gamma}_{2}(x,Q^2,P^2) \qquad (2.4)$$

where $x=Q^2/2p.q$. Now, F^{γ}_{2} and F^{γ}_{L} can be written as linear combinations of the eight independent structure functions introduced, for example, by Brown and Muzinich [18]. They decompose $W_{\mu\nu\alpha\beta}$ into the structure functions $A_1,\ldots A_8$ as follows,

$$W_{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{8} I^{i}_{\mu\nu\alpha\beta} A_i \; , \qquad (2.5)$$

where $I^{i}_{\mu\nu\alpha\beta}$ are independent tensors given in ref.[18]. For $P^2 \ll Q^2$ we find

$$F_2^\gamma = \frac{1}{2}\,(p\cdot q)q^2(A_2+A_3+2A_7+2A_8)$$

$$F_1^\gamma = -(p\cdot q)^2(A_2+A_3+2q^2A_5+2A_7+2A_8)$$

$$F_L^\gamma = -(p\cdot q)q^2A_5 \tag{2.6}$$

and thus $xF_1^\gamma=F_2^\gamma-F_L^\gamma$. Incidentally, $F_2^\gamma-F_L^\gamma$ can be written in terms of transverse helicity amplitudes [9,18,19],

$$F_2^\gamma - F_L^\gamma = 8\pi\alpha\ x[W_{++}^{++} + W_{+-}^{+-}] \tag{2.7}$$

where $W_{++}^{++}(W_{+-}^{+-})$ is the amplitude for helicities $++\to++(+-\to+-)$.

Applying the OPE to the current product $J_\mu(x)J_\nu(0)$ we get the moments of the structure functions F_2^γ,

$$\int_0^1 dx\ x^{n-2}\ F_2^\gamma(x,Q^2,P^2) =$$

$$= \sum_{i=\psi,G,NS} C_n^i(Q^2/\mu^2,g(\mu),\alpha)\langle\gamma(p)|O_n^i(\mu)|\gamma(p)\rangle$$

$$+\ C_n^\gamma(Q^2/\mu^2,g(\mu),\alpha)\langle\gamma(p)|O_n^\gamma(\mu)|\gamma(p)\rangle\ , \tag{2.8}$$

where O_n^i, C_n^i are the composite operators and their coefficient functions appearing in the OPE. ψ, G, NS and γ stand for singlet fermion, gluon, non-singlet fermion, and photon respectively. Mutatis mutandis, a similar formula holds for F_L^γ. We neglect quark masses throughout, appropriate to our $P^2, Q^2 \to \infty$ limit. (These effects have been considered by Hill and Ross, who also discussed $P^2 \neq 0$ [27]).

The essential feature in (2.8) is, of course, the appearance of photon operators O_n^γ in addition to the familiar hadronic operators.

We can freely choose the renormalization point μ, since the left-hand side of (2.8) does not depend on it. We later take $\mu^2 = -p^2 = P^2$ for the sake of convenience. For $-p^2 = P^2 >> \Lambda^2$ we can calculate the photon matrix elements of the hadronic operators perturbatively. Choosing μ^2 to be close to P^2 we get, to lowest order,

$$\langle \gamma(p) | O_n^i(\mu) | \gamma(p) \rangle = \frac{e^2}{16\pi^2} \left(\frac{-1}{2} K_i^{o,n} \ln \frac{P^2}{\mu^2} + A_n^{(2)i} \right) \tag{2.9}$$

($i = \psi, G, NS$) where $K_i^{0,n} = (\vec{K}_n^0)_i$ are one loop anomalous dimension matrices between the photon and hadronic operators. (See Appendix A).

The $A_n^{(2)i}$ depend on the renormalization scheme for the operators $O_n^i(\mu)$ [20]. This scheme dependence is cancelled by that of other terms in the expansion. For definiteness, we will work in the MS scheme. Putting $\mu^2 = -p^2 = P^2$, we find

$$\langle \gamma(p) | O_n^i(\mu) | \gamma(p) \rangle \Big|_{\mu^2 = +P^2} = \frac{e^2}{16\pi^2} A_n^{(2)i} \ . \tag{2.10}$$

The right hand side is in general non-zero.

Noting that to lowest order in the QED coupling, $\langle\gamma(p)|O_n^\gamma(\mu)|\gamma(p)\rangle=1$, we find that the moments (2.8) are given by

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2,P^2) = \sum_i C_n^i(Q^2/P^2,\bar{g}(P^2),\alpha)\, \frac{e^2}{16\pi^2} A_n^{(2)i} + C_n^\gamma(Q^2/P^2,\bar{g}(P^2),\alpha)\ . \tag{2.11}$$

III. VIRTUAL PHOTON STRUCTURE FUNCTIONS: NONLEADING EFFECTS

In this section, we calculate the moments of the virtual photon structure functions by evaluating the right hand side of (2.11).

Solving the renormalization group equation for the coefficient function to lowest order in $\alpha=e^2/4\pi$, we find

$$\begin{aligned} C_n^\gamma(Q^2/P^2,\bar{g}(P^2),\alpha) &= \\ &= \vec{X}_n(Q^2/P^2,\bar{g}(P^2),\alpha)\vec{C}_n(1,\bar{g}(Q^2)) \\ &\quad + C_n^\gamma(1,\bar{g}(Q^2),\alpha)\ , \end{aligned} \tag{3.1}$$

where

$$\vec{X}_n(Q^2/P^2,\bar{g}(P^2),\alpha) =$$

$$= \int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg' \; \frac{\vec{K}_n(g',\alpha)}{\beta(g')} \; T \exp\left[\int_{\bar{g}(Q^2)}^{g'} dg'' \; \frac{\hat{\gamma}_n(g'')}{\beta(g'')}\right] \tag{3.2}$$

with $\hat{\gamma}_n$ and $\vec{K}_n$ the usual hadronic anomalous dimension matrix and the off-diagonal element which represents the mixing between the photon operator and the hadronic operators (as given in (A.4)). $\vec{C}_n(C_n^\gamma)$ is the coefficient function of the hadronic (photon) operators.

$$\vec{C}_n(1,\bar{g}(Q^2)) = \begin{pmatrix} \delta_\psi \; (1+ \dfrac{\bar{g}^2(Q^2)}{16\pi^2} \; B_\psi^n) \\ \delta_\psi \; \dfrac{\bar{g}^2(Q^2)}{16\pi^2} \; B_G^n \\ \delta_{NS} \; (1 + \dfrac{\bar{g}^2(Q^2)}{16\pi^2} \; B_{NS}^n) \end{pmatrix} \tag{3.3}$$

$$C_n^\gamma(1,\bar{g}(Q^2),\alpha) = \frac{e^2}{16\pi^2} \; \delta_\gamma \; B_\gamma^n \tag{3.4}$$

with $\delta_\psi = \langle e^2 \rangle$, $\delta_{NS}=1$, $\delta_\gamma = 3f\langle e^4 \rangle$ (our normalization of the structure function differs from that in [7] by a factor e^2). There is no α dependence of $\vec{C}_n$ to this order. The one-loop coefficient functions $B_\psi^n = B_{NS}^n$ and B_G^n are given in ref. [20]. B_γ^n is related to B_G^n by $B_\gamma^n = (2/f)B_G^n$.

The $\vec{X}_n$ can be calculated in a straightforward way to be

$$\vec{X}_n(Q^2/P^2,\bar{g}(P^2),\alpha) =$$

$$= \frac{1}{16\pi^2}\frac{e^2}{2\beta_0}\Bigg[\vec{K}^0_n \sum_i P^n_i \frac{1}{1+\lambda^n_i/2\beta_0}\frac{16\pi^2}{\bar{g}^2(Q^2)}\left\{1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0+1}\right\}$$

$$-\vec{K}^0_n \frac{\beta_1}{\beta_0}\left\{\sum_i P^n_i \frac{\lambda^n_i/2\beta_0}{1+\lambda^n_i/2\beta_0}\left(1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0+1}\right)\right.$$

$$\left.+\sum_i P^n_i \frac{1-\lambda^n_i/2\beta_0}{\lambda^n_i/2\beta_0}\left(1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0}\right)\right\}$$

$$-\vec{K}^0_n \sum_{i,j}\frac{P^n_i\hat{\gamma}^{(1)}_n P^n_j}{2\beta_0+\lambda^n_i-\lambda^n_j}\left\{\frac{1}{\lambda^n_j/2\beta_0}\left(1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_j/2\beta_0}\right)\right.$$

$$\left.-\frac{1}{1+\lambda^n_i/2\beta_0}\left(1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0+1}\right)\right\}$$

$$+\vec{K}^{(1)}_n \sum_i P^n_i \frac{1}{\lambda^n_i/2\beta_0}\left(1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0}\right)\Bigg] \qquad (3.5)$$

where λ^n_i are the eigenvalues of the one-loop anomalous dimension matrix $\hat{\gamma}^0_n$, and P^n_i are the corresponding projection operators. $\hat{\gamma}^{(1)}_n$ is the (hadronic) two-loop anomalous dimension matrix, and $\vec{K}^0_n$ ($\vec{K}^{(1)}_n$) is the one- (two-) loop off diagonal element of the anomalous dimension matrix. (See Appendix A).

Putting all this together into the eq.(2.11) we get our final expression for the moments of F_2^γ. They are

$$\int_0^1 dx\, x^{n-2}\, F_2^\gamma(x,Q^2,P^2) =$$

$$= \frac{1}{16\pi^2}\frac{e^2}{2\beta_0}\left[\sum_i \tilde{P}_i^n \frac{1}{1+\lambda_i^n/2\beta_0}\frac{16\pi^2}{\bar{g}^2(Q^2)}\left\{1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda_i^n/2\beta_0+1}\right\}\right.$$

$$+\sum_i A_i^n\left\{1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda_i^n/2\beta_0}\right\}$$

$$+\sum_i B_i\left\{1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda_i^n/2\beta_0+1}\right\}$$

$$\left.+\, C_\gamma^n\right] \tag{3.6}$$

where i runs now over +,- and NS. In eq. (3.6) we have defined

$$\tilde{P}_i^n = \vec{K}_n^0\, P_i^n\, \vec{C}_n(1,0). \tag{3.7}$$

The coefficients A, B, C are given by

$$A_i^n = -\vec{K}_n^0 \sum_j \frac{P_j^n \gamma_n^{(1)} P_i^n}{2\beta_0+\lambda_j^n-\lambda_i^n}\vec{C}_n(1,0)\frac{1}{\lambda_i^n/2\beta_0}$$

$$-\vec{K}_n^0\frac{\beta_1}{\beta_0}P_i^n\,\vec{C}_n(1,0)\frac{1-\lambda_i^n/2\beta_0}{\lambda_i^n/2\beta_0}$$

$$+\vec{K}_n^{(1)}P_i^n\,\vec{C}_n(1,0)\frac{1}{\lambda_i^n/2\beta_0}$$

$$-2\beta_0\,\vec{A}_n^{(2)}\,P_i^n\,\vec{C}_n(1,0) \tag{3.8}$$

$$B_i^n = \vec{K}_n^0 \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{2\beta_0+\lambda_i^n-\lambda_j^n} \vec{C}_n(1,0) \frac{1}{1+\lambda_i^n/2\beta_0}$$

$$+\vec{K}_n^0 P_i^n \begin{pmatrix} \delta_\psi B_\psi^n \\ \delta_\psi B_{G_n}^n \\ \delta_{NS} B_{NS}^n \end{pmatrix} \frac{1}{1+\lambda_i^n/2\beta_0}$$

$$-\vec{K}_n^0 \frac{\beta_1}{\beta_0} P_i^n \vec{C}_n(1,0) \frac{\lambda_i^n/2\beta_0}{1+\lambda_i^n/2\beta 0} \tag{3.9}$$

$$C_\gamma^n = 2\beta_0(\delta_\gamma B_\gamma^n + \vec{A}_n^{(2)}\vec{C}_n(1,0))$$

$$= 24\beta_0 f \langle e^4\rangle \left[\frac{6}{n+1} - \frac{2}{n} - \frac{6}{n+2} + \frac{2}{n^2} - \frac{4}{(n+1)^2} + \frac{4}{(n+2)^2}\right] \tag{3.10}$$

(Explicit expressions for A and B will be found in Appendix B)

Equation (3.6) is the main result of the present paper. The first term is the leading log result obtained in a different way in ref. [11]. (Of course, $\bar{g}(Q^2)$ is now to be taken as the effective coupling including two-loop terms.)

The remaining terms in (3.6) are nonleading QCD corrections. Note that there is no unknown term in (3.6), so that this equation is also valid for n=2 provided we regard the expression $(1/\varepsilon)(1-x^\varepsilon)$ as its limiting value for $\varepsilon\to 0$, $-\ln x$.

We now discuss the renormalization scheme independence of A_i^n, B_i^n and C_γ^n. We start with B_i^n; it can be written in terms of a scheme-independent combination of two-loop anomalous dimensions and one-loop coefficient functions in the hadronic sector ([28], [29]; see also Appendix B). Using the scheme-independent coefficients $R_{2,n}^i$ introduced in [28], we write

$$B_i^n = L_i^n \, R_{2,n}^i \quad (i=+,-,NS) \tag{3.11}$$

where

$$L_i^n = \tilde{P}_i^n \frac{1}{1+\lambda_i^n/2\beta_0} \tag{3.12}$$

The scheme independence of B_i^n follows immediately from these two equations.

As for C_γ^n, we note that

$$\delta_\gamma \, B_\gamma^n = 6\langle e^4\rangle B_G^n \quad , \tag{3.13}$$

$$\vec{A}_n^{(2)} \, \vec{C}_n(1,0) = 6\langle e^4\rangle \, A_{n\,G}^{(2)\psi} \quad . \tag{3.14}$$

Thus, from (3.10),

$$C_\gamma^n = 2\beta_0 \cdot 6\langle e^4\rangle \, (B_G^n + A_{n\,G}^{(2)\psi}). \tag{3.15}$$

(In these expressions, $A_{n\,G}^{(2)\psi}$ differs from that in ref. [20]

by a factor $g^2/16\pi^2$; of course, all this is written in the MS scheme.) Now, from ref. [20], we know that $B^n_G + A^{(2)\psi}_{n\,G}$ is scheme independent. Thus, C^n_γ is also scheme independent.

The scheme independence of the A^n_i follows from (3.8) and also (3.6).

Now we go on to the renormalization scheme dependence of the QCD coupling. The relation between the MS and $\overline{\text{MS}}$ schemes is

$$\Lambda_{\overline{MS}} = \Lambda_{MS} \exp\left[\frac{1}{2}\,(\ln\,4\pi - \gamma_E)\right] \tag{3.16}$$

where γ_E=0.5772... and

$$\alpha_S^{MS}(Q^2) = \alpha_S^{\overline{MS}}(Q^2) - \frac{[\alpha_s^{\overline{MS}}(Q^2)]^2}{4\pi}\,\beta_0(\ln 4\pi - \gamma_E) \tag{3.17}$$

Then, under the MS$\to\overline{\text{MS}}$ transformation we have

$$A^n_i \to \bar{A}^n_i = A^n_i + \tilde{P}^n_i\beta_0(\ln\,4\pi\,-\,\gamma_E) \tag{3.18}$$

$$B^n_i \to \bar{B}^n_i = B^n_i - L^n_i\beta_0(\ln\,4\pi\,-\,\gamma_E) \quad , \tag{3.19}$$

where $A^n_i(\bar{A}^n_i)$ and $B^n_i(\bar{B}^n_i)$ are in the MS($\overline{\text{MS}}$) scheme.

$\bar{A}^n_i$ and $\bar{B}^n_i$ are obtained by replacing the MS scheme quantities B^n_ψ, B^n_G, B^n_γ, $\tilde{A}^{(2)}_n$ by the $\overline{\text{MS}}$ scheme correlates, $\bar{B}^n_\psi$, etc. These can be got by removing the term proportional to $\ln 4\pi - \gamma_E$ as follows:

$$\bar{B}^n_\psi = B^n_\psi - \frac{1}{2}\gamma^{0,n}_{\psi\psi}(\ln 4\pi - \gamma_E) \tag{3.20}$$

$$\bar{B}^n_G = B^n_G - \frac{1}{2}\gamma^{0,n}_{\psi G}(\ln 4\pi - \gamma_E) \tag{3.21}$$

$$\bar{B}^n_\gamma = B^n_\gamma - \frac{1}{f}\gamma^{0,n}_{\psi G}(\ln 4\pi - \gamma_E) \tag{3.22}$$

$$\vec{\bar{A}}^{(2)}_n = \vec{A}^{(2)}_n - \frac{1}{2}\vec{K}^0_n(\ln 4\pi - \gamma_E) \tag{3.23}$$

From (3.21) and the relation

$$\bar{A}^{(2)\psi}_{n\ G} = A^{(2)\psi}_{n\ G} + \frac{1}{2}\gamma^{0,n}_{\psi G}(\ln 4\pi - \gamma_E), \tag{3.24}$$

we see that C^n_γ (eq.(3.15)) does not change under $MS\to\overline{MS}$.

Incidentally, we can now rewrite (3.6) in a slightly different form. Introducing

$$d^n_i = \lambda^n_i/2\beta_0 \qquad (i=+,-,NS) \quad , \tag{3.25}$$

and using (3.11) and (3.12), we get

$$\int_0^1 dx\ x^{n-2}F^\gamma_2(x,Q^2,P^2) =$$

$$= \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\Bigg[\sum_i L^n_i \frac{4\pi}{\alpha_s(Q^2)}\left(1 + \frac{\alpha_s(Q^2)}{4\pi}R^i_{2,n}\right) \times\left\{1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{1+d^n_i}\right\} + \sum_i A^n_i\left\{1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d^n_i}\right\} + C^n_\gamma\Bigg] \tag{3.26}$$

This is useful because the quantities $1+(\alpha_s(Q^2)/4\pi)R^i_{2,n}$ in the first term are the same ones we meet with in the case of the nucleon structure function [28,29].

For completeness, we now give the formula for the moments of the longitudinal structure function F^γ_L,

$$\int_0^1 dx\ x^{n-2} F^\gamma_L(x,Q^2,P^2) =$$

$$= \vec{X}_n(Q^2/P^2,\bar{g}(P^2),\alpha)\vec{C}^{(L)}_n(1,\bar{g}(Q^2))$$

$$+ C^{(L)\gamma}_n(1,\bar{g}(Q^2),\alpha) \tag{3.27}$$

$$= \frac{e^2}{16\pi^2}\frac{1}{2\beta_0}\left[\sum_i \tilde{P}^{(L),n}_i \frac{1}{1+\lambda^n_i/2\beta_0}\left\{1-\left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(P^2)}\right)^{\lambda^n_i/2\beta_0+1}\right\} + 2\beta_0\delta_\gamma B^n_{\gamma,L}\right]$$

where

$$\tilde{P}^{(L),n}_i = \vec{K}^0_n P^n_i \begin{pmatrix} \delta_\psi B^n_{\psi,L} \\ \delta_\psi B^n_{G,L} \\ \delta_{NS} B^n_{NS,L} \end{pmatrix} \tag{3.28}$$

$$B^n_{\psi,L} = B^n_{NS,L} = \frac{4}{3}\frac{4}{n+1}$$

$$B^n_{G,L} = \frac{8f}{(n+1)(n+2)}$$

$$B^n_{\gamma,L} = \frac{2}{f} B^n_{G,L}$$

(Expressions for $\tilde{P}^{(L),n}_i$ and $\delta_\gamma B^n_{\gamma,L}$ will be found in Appendix C.)

IV. REAL PHOTON STRUCTURE FUNCTIONS

Starting from our final result (3.6), we can now recover the moments of the real photon structure function. These follow formally by setting $P^2=\Lambda^2$,

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2) =$$

$$= \frac{e^2}{16\pi^2}\frac{1}{2\beta_0}\left[\sum_{i=+,-,NS} \tilde{P}^n_i \frac{1}{1+\lambda^n_i/2\beta_0} \frac{16\pi^2}{\bar{g}^2(Q^2)} + \sum_{i=+,-,NS} A^n_i + \sum_{i=+,-,NS} B^n_i + C^n_\gamma\right] \qquad (4.1)$$

However, this equation no longer holds for n=2. There is an extra term proportional to a Kronecker δ-function $\delta_{n,2}$. Its coefficient is an unknown constant coming from the photon matrix element of the hadronic energy-momentum tensor. In order to see the situation more clearly,* consider the usual expression for the moments of F_2^γ (including the hadronic contribution) [3,7],

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*We thank W.A. Bardeen for discussions on this point, and advice on this section.

$$M_n^\gamma(Q^2) \equiv \int_0^1 dx\; x^{n-2} F_2^\gamma(x,Q^2) =$$

$$= \frac{\alpha}{4\pi}\,\frac{1}{2\beta_0}\left[\frac{4\pi}{\alpha_S(Q^2)}\,a_n + b_n + \sum_{i=+,-,NS} M_n^i\,[\alpha_S(Q^2)]^{d_i^n}\right], \tag{4.2}$$

where

$$a_n = \sum_i L_i^n \tag{4.3}$$

$$b_n = \sum_i A_i^n + \sum_i B_i^n + C_\gamma^n \tag{4.4}$$

If we consider the $M_n^\gamma(Q^2)$ to be a function of the continuous variable n, then (4.4) requires b_n to have a singularity at n=2. This singularity has to be cancelled by one in M_n^-, so as to give the correct $M_2^\gamma(Q^2)$.

Explicitly, we have

$$b_n \xrightarrow[n\to 2]{} \frac{c}{d_-^n} + b_2' \tag{4.5}$$

where $d_-^2=0$. Thus, we require

$$M_n^- \xrightarrow[n\to 2]{} -\frac{c}{d_-^n} + b_2'' \tag{4.6}$$

The constants c and b_2' are, of course, perturbatively calculable,* but b_2'' is not.

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* $c = \lim_{n\to 2} b_n d_-^n = -(\beta_1/\beta_0\lambda_+)K_\psi^0\gamma_{GG}^0\langle e^2\rangle$ where the index n=2 has been omitted and use has been made of the relations: $\psi_{\psi\psi}^{0,(1)} = -\psi_{G\psi}^{0,(1)}, \psi_{\psi G}^{0,(1)} = -\gamma_{GG}^{0,(1)}$ and $K_\psi^{(1)} = -K_G^{(1)}$.

Thus,

$$M_n^{\gamma}(Q^2) \underset{n\to 2}{\longrightarrow} \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\left[\frac{4\pi}{\alpha_S(Q^2)}a_2 - c\ln\alpha_S(Q^2) + b'\right] \tag{4.7}$$

where $b'=b_2'+b_2''$ is an unknown constant. This expression for n=2 was first obtained by Witten [3].

The singularity in M_n^- can at least be isolated by introducing a normalization point Q_0^2 (as is done for the nucleon target case). Taking $\mu^2=Q_0^2>>\Lambda^2$ we get

$$\begin{aligned} M_n^{\gamma}(Q^2) = \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\Bigg[& \sum_i L_i^n \frac{4\pi}{\alpha_S(Q^2)} + \\ & + \sum_i A_i^n + \sum_i B_i^n + C_\gamma^n \\ & + \sum_i \left\{2\beta_0\, \tilde{A}_n^i(Q_0^2) - 4\pi\frac{1}{\alpha_S(Q_0^2)} L_i^n - A_i^n\right\}\left(\frac{\alpha_S(Q^2)}{\alpha_S(Q_0^2)}\right)^{d_i^n} \\ & + O(\alpha_S)\Bigg] \end{aligned} \tag{4.8}$$

where

$$\tilde{A}_n^i(Q_0^2) = [\vec{A}_n^{\gamma}(Q_0^2) - \vec{A}_n^{(2)}]P_i^n\vec{C}_n(1,0) \tag{4.9}$$

with

$$\langle\gamma|\vec{O}_n(\mu)|\gamma\rangle = \frac{\alpha}{4\pi}\vec{A}_n^{\gamma}(\mu^2) \tag{4.10}$$

being the uncalculable matrix element.

In eq. (4.8) we have neglected $O(\alpha_S)$ terms. For consistency, we should therefore keep only log decreasing terms with $d_i^n<1$.

The unknown hadronic components M_n^i in (4.2) are related to (4.9) as follows,

$$M_n^i = 2\beta_0\tilde{A}_n^i(Q_0^2)[\alpha_S(Q_0^2)]^{-d_i^n} - 4\pi L_i^n[\alpha_S(Q_0^2)]^{-1-d_i^n} - A_i^n[\alpha_S(Q_0^2)]^{-d_i^n} \tag{4.11}$$

The singularity we are looking for is thus in A_i^n. (This situation is similar to that for a nucleon target, where a singularity appears in a quantity called $\bar{A}_n^+$ by Bardeen and Buras [28].)

Taking Q^2 very large and neglecting logarithmically decreasing terms we finally get the moments

$$M_n^\gamma(Q^2) \to \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\left[\frac{4\pi}{\alpha_S(Q^2)}\sum_i L_i^n + \sum_i A_i^n + \sum_i B_i^n + C_\gamma^n + \delta_{n,2}\{-c \ln \alpha_S(Q^2) + \tilde{b}_2 - A_-^2\}\right] \tag{4.12}$$

where $\tilde{b}_2=2\beta_0\tilde{A}_-^2(Q_0^2)+c\ \ln\alpha_s(Q_0^2)$. This now isolates the additional Kronecker δ-function term. When inverted, it gives rise to a δ-function at x=0. (Of course, this δ-function only appears for $Q^2=\infty$; at finite Q^2 it will be smeared out.)

This is the prediction that perturbative QCD makes for the real photon structure function at low x. There is a term which behaves as -const./x, giving a negative structure function, and a δ-function at x=0. To take account of the finite Q^2 smearing of the δ-function, we would have to know $M_n^i(Q_0^2)$ or $\tilde{A}_n^i(Q_0^2)$ as a function of n. These cannot be calculated in perturbation theory.

V. INVERSION OF THE MOMENTS AND RESULTS

Numerical inversion of the moments is now standard. The integrand has to be analytically continued to complex n. For the two-loop anomalous dimensions [21-26] and the one-loop coefficient functions [20], we adopted the asymptotic expansion form obtained by Gonzalez-Arroyo, Lopez and Yndurain [22,23]. (This form was, however, modified slightly so as to fit the low moments.) Their form has simple analytic properties and reproduces the moments quite well. The numerical results are independent of the location of the contour (of course, we constrained it to be to the right of all singularities).

Our results for the virtual photon structure function $F_2^\gamma(x,Q^2,P^2)$ and for the real photon $F_2^\gamma(x,Q^2)$ are shown in Figs. 3a and 3b. We chose Q^2=30 GeV^2 and P^2=1 GeV^2 with

$\Lambda_{\overline{MS}}$=500 MeV (Fig.3a) and 100 MeV (Fig.3b).† We also show our previous leading-log calculation for completeness [11]. Here, in our present analysis, we have taken f=4.

From the figures it is clear that the nonleading corrections appreciably modify the virtual photon structure function--especially at large x. We also confirm that the (perturbatively calculated) real photon structure function is negative at small x. This was first pointed out by Duke and Owens [8]. (Remember, however, that we do not attempt to smear out the $\delta(x)$ contribution.)

Where does the negative structure function come from? As given in (4.4), b_n to nonleading order is the sum of seven parameters A_i^n, B_i^n and C_γ^n (for i=+,-,NS). We list them for n=2,4,... 20 in table 1. One might at first think that the negative structure function is due to large negative values of C_γ^n. This is not the case, however. Defining D_γ^n and its inverse transform $D_\gamma(x)$ such that

$$D_\gamma^n = C_\gamma^n / 24\beta_0 f\langle e^4\rangle$$
$$D_\gamma^n = \int_0^1 dx\ x^{n-2} D_\gamma(x) \qquad (5.1)$$

Then we easily find

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†Strictly speaking, we ought to modify x to a ξ scaling variable [14]. This is however, a part of the P^2/Q^2 corrections we ignore.

$$D_\gamma(x) = 6x^2(1-x) - 2x - 2x(1-2x+2x^2)\ln x \quad , \tag{5.2}$$

which we plotted in Fig. 4a. We see from this figure that C^n_γ mainly contributes at large x, reducing the leading-log results.

In fact, the negative structure function arises from A^n_-. This can be seen by numerically inverting A^n_-, as shown in Fig. 4b, where $A^n_-=\int_0^1 dx\ x^{n-2}A_-(x)$. (For a virtual photon, the effect of A^n_- is reduced due to a factor

$$1 - \left(\frac{\alpha_S(Q^2)}{\alpha_S(P^2)}\right)^{d^n_-}$$

which multiplies it.) The negative structure function thus appears due to the same term A^n_- which developed a singularity for n=2 in the real photon case.* As we remarked before, there is no trouble for n=2 and a virtual photon target in the kinematic limit (1.1). Correspondingly, F^γ_2 is well-behaved as $x\to 0$ even for P^2=1 GeV^2 in Figs. 3a,3b.

In Fig. 5 we show the longitudinal structure function $F^\gamma_L(X,Q^2,P^2)$ for Q^2=30 GeV^2, P^2=1 GeV^2 and P^2=4 GeV^2 with $\Lambda_{\overline{MS}}$=100 MeV. For comparison, we also plot $F^\gamma_L(x,Q^2)$, the longitudinal structure function.

The statement we made in our paper on the leading log calculation [11], that the off-shell photon structure functions approach the box contribution, also holds in the

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*Some further discussion for the real photon case is given by W.A. Bardeen, talk given at Lepton-Photon Symposium (Bonn,1981).

present case. To see this, we need the box contribution including finite terms. Including also nonleading pieces, these are

$$F_2^\gamma(x,Q^2,P^2)/\langle e^4\rangle 3f\,\frac{\alpha}{\pi}\ln Q^2/P^2 =$$

$$= x[x^2+(1-x)^2] \tag{5.3}$$

$$-\frac{1}{\ln Q^2/P^2}\,2x[1-3x+3x^2+(1-2x+2x^2)\ln x]$$

$$F_L^\gamma(x,Q^2,P^2)/\langle e^4\rangle 3f\,\frac{\alpha}{\pi} = 4x^2(1-x) \tag{5.4}$$

where we neglect power corrections P^2/Q^2, and quark mass effects. Note that the second term in (5.3) is just proportional to the inversion of C_γ^n.

It is not hard to see that in the limit $\ln Q^2/P^2 \ll \ln P^2/\Lambda^2$ the moments of F_2^γ and F_L^γ approach the box contribution ones as follows,

$$\begin{aligned} M_{2,n}^\gamma(Q^2,P^2) &\to \frac{e^2}{16\pi^2}\frac{1}{2\beta_0}\left[\sum_i \tilde{P}_i^n\beta_0\ln\frac{Q^2}{P^2}+C_\gamma^n\right] \\ &= 3f\,\frac{\alpha}{\pi}\langle e^4\rangle\left[\frac{n^2+n+2}{n(n+1)(n+2)}\ln\frac{Q^2}{P^2}+D_\gamma^n\right] \end{aligned} \tag{5.5}$$

$$M_{L,n}^\gamma(Q^2,P^2) \to 3f\,\frac{\alpha}{\pi}\langle e^4\rangle\frac{4}{(n+1)(n+2)} \tag{5.6}$$

We show this numerically in Fig. 6 for $F_2^\gamma(x,Q^2,P^2)$ and in Fig. 5 for $F_L^\gamma(x,Q^2,P^2)$.

VI. DISCUSSION AND CONCLUSION

In this paper we have studied the structure functions of a virtual photon in the kinematic region where P^2 (the target mass squared) and Q^2 diverge with P^2/Q^2 small. The calculation includes nonleading log QCD effects. However, all effects proportional to powers of P^2/Q^2 are neglected. We find that the nonleading log corrections to our previous leading log results[11] are important. However, the general conclusion that the structure function approaches the pointlike box (or Born) contribution for $P^2 \neq 0$ remains unchanged. This is because the box diagram also has a significant nonleading log piece. The real photon structure function (or, rather, the nonleading log calculation of it) has a slight pathology in that it is negative for small x. This is, of course, the region where the (ignored) hadronic contribution to the structure function is important. However, it is interesting that even for $P^2=1$ GeV^2, we find a perturbative structure function which is positive at all x. For large enough P^2 we would expect this, since the entire structure function is perturbatively calculable. There is no unknown hadronic piece (which contributed to the n=2 moment for real photons). However, it is interesting that the difficulty with a perturbative calculation at low x is confined to target photons very near the mass shell. We expect that this is in some sense the region where confinement effects are most important.

We have ignored $(P^2/Q^2)^k$ $(k=1,2,\ldots)$ corrections. Formally, these are target mass and higher twist contributions. For a real photon target mass corrections are trivially absent. Higher twist contributions are expected to involve a target radius so that for the pointlike photon term they should start with the order of magnitude $\alpha_s(Q^2)\Lambda^2/Q^2$. This should be small if Λ^2/Q^2 is small. Real photon targets may have small higher twist corrections. Off-shell, the $(P^2/Q^2)^k$ power corrections should in principle be calculable. (The box diagram $(P^2/Q^2)^k$ terms have been calculated by Frazer and Gunion [13].) Because of this, it may be useful to comment briefly on an extreme limit, double deep inelastic scattering with P^2 and Q^2 both large and also $W^2=(p^2+q)^2\to\infty$ [30].* In this case, the leading term is given by the box diagram. There are no large logarithms, since $[\alpha_s(Q^2)\ln(Q^2/P^2)]^n$ is always small for fixed P^2/Q^2. Gluon radiative corrections are nonleading, and the now important power corrections are entirely calculable. If we consider the Nachtmann moments (or any other) for $\gamma^*(P)+\gamma^*(q)\to$ hadrons and also those for $\gamma^*(p)+\gamma^*(q)\to\mu^+\mu^-$ and take the ratio of one to the other, we have the very simple result

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*Chase [31] has recently studied QCD radiative corrections in the region where W^2 is fixed.

$$R_n \equiv \frac{M_n(2\gamma\to\text{hadrons})}{M_n(2\gamma\to\mu^+\mu^-)} = 3\Sigma Q_i^4(1+C_n\,\frac{\alpha_S(Q^2)}{\pi} + \ldots)$$

where only $C_n(P^2,Q^2)$ remains to be calculated. Thus in this kinematic region the P^2/Q^2 corrections pose no problem. It would be useful to extend the analysis to small P^2/Q^2, where additional logarithms may become important.

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APPENDIX

A. Notation

Here we summarize our notation which we adopt from ref.[7].

The renormalization-group equation for coefficient functions reads to lowest order in α as follows

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\begin{pmatrix} \vec{C}_n(\frac{Q^2}{\mu^2}, g(\mu)) \\ \\ C_n^{\gamma}(\frac{Q^2}{\mu^2}, g(\mu), \alpha) \end{pmatrix} = \gamma_n(g,\alpha)\begin{pmatrix} \vec{C}_n(\frac{Q^2}{\mu^2}, g(\mu)) \\ \\ C_n^{\gamma}(\frac{Q^2}{\mu^2}, g(\mu), \alpha) \end{pmatrix} \tag{A.1}$$

where

$$\vec{C}_n(\frac{Q^2}{\mu^2}, g(\mu)) = \begin{pmatrix} C_n^{\psi}(Q^2/\mu^2, g(\mu)) \\ C_n^{G}(Q^2/\mu^2, g(\mu)) \\ C_n^{NS}(Q^2/\mu^2, g(\mu)) \end{pmatrix} \tag{A.2}$$

The β-function is expanded as

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0 - \frac{g^5}{(16\pi^2)^2}\beta_1 + O(g^7) \tag{A.3}$$

with $\beta_0 = 11-(2/3)f$, $\beta_1 = 102-(38/3)f$ and f being the number of flavors.

The anomalous dimension matrix can be taken to the presently concerned order as

$$\gamma_n(g,\alpha) = \begin{pmatrix} \hat{\gamma}_n(g) & 0 \\ \vec{K}_n(g,\alpha) & 0 \end{pmatrix} \tag{A.4}$$

where $\hat{\gamma}_n(g)$ is the usual anomalous dimension in the hadronic sector

$$\hat{\gamma}_n(g) = \begin{pmatrix} \gamma^n_{\psi\psi}(g) & \gamma^n_{G\psi}(g) & 0 \\ \gamma^n_{\psi G}(g) & \gamma^n_{GG}(g) & 0 \\ 0 & 0 & \gamma^n_{NS}(g) \end{pmatrix} \tag{A.5}$$

$\vec{K}_n(g,\alpha)$ stands for the mixing between hadronic and photon operators

$$\vec{K}_n(g,\alpha) = (K^n_\psi(g,\alpha),\ K^n_G(g,\alpha),\ K^n_{NS}(g,\alpha)) \tag{A.6}$$

The anomalous dimensions are expanded as

$$\hat{\gamma}_n(g) = \frac{g^2}{16\pi^2}\,\hat{\gamma}^0_n + \frac{g^4}{(16\pi^2)^2}\,\hat{\gamma}^{(1)}_n + O(g^6) \tag{A.7}$$

$$\vec{K}_n(g,\alpha) = -\frac{e^2}{16\pi^2}\,\vec{K}^0_n - \frac{e^2g^2}{(16\pi^2)^2}\,\vec{K}^{(1)}_n + O(e^2g^4) \tag{A.8}$$

The one-loop anomalous dimension matrix $\hat{\gamma}^0_n$ can be expressed in terms of the eigenvalues $\lambda^n_i(i=+,-,NS)$ as

$$\hat{\gamma}^0_n = \sum_{i=+,-,NS} \lambda^n_i P^n_i \tag{A.9}$$

where P^n_i are corresponding projection operators

$$P_{\pm}^{n} = \frac{1}{\lambda_{\pm}^{n}-\lambda_{\mp}^{n}} \begin{pmatrix} \gamma_{\psi\psi}^{0,n}-\lambda_{\mp}^{n}, & \gamma_{G\psi}^{0,n} & , & 0 \\ \gamma_{\psi G}^{0,n}, & \gamma_{GG}^{0,n}-\lambda_{\mp}^{n}, & & 0 \\ 0 \quad , & 0 & , & 0 \end{pmatrix} \tag{A.10}$$

$$P_{NS}^{n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.11}$$

$$\lambda_{\pm}^{n} = \frac{1}{2}\{\gamma_{\psi\psi}^{0,n} + \gamma_{GG}^{0,n} \pm [(\gamma_{\psi\psi}^{0,n} - \gamma_{GG}^{0,n})^{2} + 4\gamma_{\psi G}^{0,n}\gamma_{G\psi}^{0,n}]^{1/2}\} \tag{A.12}$$

$$\lambda_{NS}^{n} = \gamma_{NS}^{0,n} \tag{A.13}$$

$\vec{K}_n^0$ and $\vec{K}_n^{(1)}$ are three-component row vectors

$$\vec{K}_n^0 = (K_{\psi}^{0,n},\ 0,\ K_{NS}^{0,n})$$

$$K_{\psi}^{0,n} = 24f\langle e^2\rangle \frac{n^2+n+2}{n(n+1)(n+2)} \tag{A.14}$$

$$K_{NS}^{0,n} = 24f(\langle e^4\rangle - \langle e^2\rangle^2) \frac{n^2+n+2}{n(n+1)(n+2)}$$

$$\vec{K}_n^{(1)} = (K_\psi^{(1),n},\ K_G^{(1),n},\ K_{NS}^{(1),n})$$

$$K_\psi^{(1),n} = \frac{12}{3} f\langle e^2\rangle B_n^{fg} \tag{A.15}$$

$$K_{NS}^{(1),n} = \frac{12}{3} f(\langle e^4\rangle - \langle e^2\rangle^2) B_n^{fg}$$

$$K_G^{(1),n} = -\frac{12}{3} f\langle e^2\rangle B_n^{gg}$$

where the values of B_n^{fg} and B_n^{gg} are given in ref.[21]

The running coupling constant is written in terms of the β-function (A.3) to two-loop order as

$$\frac{\bar{g}^2(Q^2)}{16\pi^2} = \frac{\alpha_S(Q^2)}{4\pi} = \frac{1}{\beta_0 \ln Q^2/\Lambda^2} - \frac{\beta_1 \ln\ln Q^2/\Lambda^2}{\beta_0^3 \ln^2 Q^2/\Lambda^2} \tag{A.16}$$

$A_n^{(2)i}$ ($i=\psi,G,NS$) in eq.(2.9) are related to $A_{n\ G}^{(2)\psi}$ in eq. (6.22) of ref.[20] as follows

$$\vec{A}_n^{(2)} = (A_n^{(2)\psi},\ A_n^{(2)G},\ A_n^{(2)NS})$$

$$= 6f(\langle e^2\rangle,\ 0,\ \langle e^4\rangle - \langle e^2\rangle^2)\tilde{A}_{n\ G}^{(2)\psi} \tag{A.17}$$

with

$$A_{n\ G}^{(2)\psi} = \frac{g^2}{16\pi^2}\tilde{A}_{n\ G}^{(2)\psi}$$

B. Explicit expression for the parameters which appear in Eq. (3.6)

We express the parameters A_i^n, B_i^n (i=+,-,NS) and C_γ^n in terms of anomalous dimensions and coefficient functions.

$$A_+^n = \langle e^2\rangle \frac{1}{\lambda_+^n(\lambda_+^n-\lambda_-^n)(2\beta_0+\lambda_-^n-\lambda_+^n)}$$

$$\times\Big[K_\psi^{0,n}\{(\gamma_{\psi\psi}^{0,n}-2\beta_0-\lambda_-^n)\gamma_{\psi\psi}^{(1),n} + \gamma_{G\psi}^{0,n}\gamma_{\psi G}^{(1),n}\}(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)$$

$$+ K_\psi^{0,n}\{(\gamma_{\psi\psi}^{0,n}-2\beta_0-\lambda_-^n)\gamma_{G\psi}^{(1),n} + \gamma_{G\psi}^{0,n}\gamma_{GG}^{(1),n}\}\gamma_{\psi G}^{0,n}$$

$$+ 2\beta_0(2\beta_0+\lambda_-^n-\lambda_+^n)\{K_\psi^{(1),n}(\gamma_{\psi\psi}^{0,n}-\lambda_-^n) + K_G^{(1),n}\gamma_{\psi G}^{0,n}\}$$

$$- 2\beta_0(2\beta_0+\lambda_-^n-\lambda_+^n)\lambda_+^n A_n^{(2)\psi}(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)$$

$$- \frac{\beta_1}{\beta_0} K_\psi^{0,n}(2\beta_0+\lambda_-^n-\lambda_+^n)(2\beta_0-\lambda_+^n)(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)\Big] \tag{B.1}$$

$$A_-^n = A_+^n(+ \leftrightarrow -) \tag{B.2}$$

$$A_{NS}^n = \frac{1}{\lambda_{NS}^n}[-K_{NS}^{0,n}\gamma_{NS}^{(1),n}+2\beta_0 K_{NS}^{(1),n}-2\beta_0\lambda_{NS}^n A_n^{(2)NS}$$

$$- \frac{\beta_1}{\beta_0} K_{NS}^{0,n}(2\beta_0-\lambda_{NS}^n)] \tag{B.3}$$

$$B_+^n = \langle e^2\rangle K_\psi^{0,n} \frac{1}{(2\beta_0+\lambda_+^n)(\lambda_+^n-\lambda_-^n)(2\beta_0+\lambda_+^n-\lambda_-^n)}$$

$$\times[\{(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)\gamma_{\psi\psi}^{(1),n} + \gamma_{G\psi}^{0,n}\gamma_{\psi G}^{(1),n}\}(2\beta_0+\gamma_{\psi\psi}^{0,n}-\lambda_-^n)$$

$$+\{(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)\gamma_{G\psi}^{(1),n} + \gamma_{G\psi}^{0,n}\gamma_{GG}^{(1),n}\}\gamma_{\psi G}^{0,n}$$

$$+ 2\beta_0(2\beta_0+\lambda_+^n-\lambda_-^n)\{(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)B_\psi^n + \gamma_{G\psi}^{0,n}B_G^n\}$$

$$- \frac{\beta_1}{\beta_0}(2\beta_0+\lambda_+^n-\lambda_-^n)\lambda_+^n(\gamma_{\psi\psi}^{0,n}-\lambda_-^n)] \tag{B.4}$$

$$B_-^n = B_+^n(+ \leftrightarrow -) \tag{B.5}$$

$$B_{NS}^n = K_{NS}^{0,n}\frac{1}{2\beta_0+\lambda_{NS}^n}(\gamma_{NS}^{(1),n}+2\beta_0 B_\psi^n - \frac{\beta_1}{\beta_0}\lambda_{NS}^n) \tag{B.6}$$

In terms of scheme-independent coefficients $R_{2,n}^i$ (i=+,-,NS) introduced in ref.[28] B_i^n can be rewritten as

$$B_i^n = L_i^n R_{2,n}^i \qquad (i=+,-,NS) \tag{B.7}$$

where

$$L_i^n = \tilde{P}_i^n \frac{1}{1+\lambda_i^n/2\beta_0} \tag{B.8}$$

with $\tilde{P}_i^n$ being the parameters which appear in eq. (3.6) and given by

$$\tilde{P}^n_+ = \frac{\gamma^{0,n}_{\psi\psi} - \lambda^n_-}{\lambda^n_+ - \lambda^n_-} \langle e^2 \rangle K^{0,n}_\psi \tag{B.9}$$

$$\tilde{P}^n_- = \tilde{P}^n_+ (+ \leftrightarrow -) \tag{B.10}$$

$$\tilde{P}^n_{NS} = K^{0,n}_{NS} \tag{B.11}$$

$R^+_{2,n}$, for example, can be written as

$$R^+_{2,n} = \frac{\gamma^{(1),n}_{++}}{2\beta_0} - \frac{\gamma^{(1),n}_{+-}}{2\beta_0 + \lambda^n_+ - \lambda^n_-} + B^+_{2,n} - \frac{\beta_1}{2\beta_0} \lambda^n_+ \tag{B.12}$$

where $\gamma^{(1),n}_{++}$ and $\gamma^{(1),n}_{+-}$ are elements of two-loop anomalous dimension matrices $\hat{\gamma}^{(1),n}$ in the basis in which $\hat{\gamma}^{0,n}$ is diagonal. $B^+_{2,n}$ is given by

$$B^+_{2,n} = B^n_\psi + \frac{\gamma^{0,n}_{G\psi}}{\gamma^{0,n}_{\psi\psi} - \lambda^n_-} B^n_G = B^n_\psi + \frac{\lambda^n_+ - \gamma^{0,n}_{\psi\psi}}{\gamma^{0,n}_{\psi G}} B^n_G \tag{B.13}$$

and similar equation for $R^-_{2,n}$ and $R^{NS}_{2,n}$.

$$C^n_\gamma = 24\beta_0 f \langle e^4 \rangle \left[\frac{6}{n+1} - \frac{2}{n} - \frac{6}{n+2} + \frac{2}{n^2} - \frac{4}{(n+1)^2} + \frac{4}{(n+2)^2} \right] \tag{B.14}$$

C. Explicit expression for the parameters in the longitudinal structure function

$$\tilde{P}_{+}^{(L),n} = \frac{\langle e^2\rangle}{\lambda_{+}^{n}-\lambda_{-}^{n}} K_{\psi}^{0,n}\left[(\gamma_{\psi\psi}^{0,n}-\lambda_{-}^{n})\frac{16}{3(n+1)} + \gamma_{G\psi}^{0,n}\frac{8f}{(n+1)(n+2)}\right] \tag{C.1}$$

$$\tilde{P}_{-}^{(L),n} = \tilde{P}_{+}^{(L),n}(+ \leftrightarrow -) \tag{C.2}$$

$$\tilde{P}_{NS}^{(L),n} = K_{NS}^{0,n}\frac{16}{3(n+1)} \tag{C.3}$$

$$\delta_{\gamma}B_{\gamma,L}^{n} = \frac{48f\langle e^4\rangle}{(n+1)(n+2)} \tag{C.4}$$

TABLE CAPTION

Table 1. Numerical values of A_i^n, B_i^n (i=+,-,NS) and C_γ^n for n=2,4,...,20. For these computations, we have adopted the $\overline{MS}$ scheme.

n	A_+^n	A_-^n	A_{NS}^n	B_+^n	B_-^n	B_{NS}^n	C_γ^n	Sum
2	8.480	-	-6.088	3.885	-8.628	1.309	-16.324	-
4	5.782	-5.789	-1.389	0.173	0.730	2.161	-18.796	-17.127
6	0.459	-3.737	-1.049	0.039	6.049	2.240	-15.931	-11.930
8	0.229	-3.184	-0.974	0.015	5.957	2.161	-13.555	-9.351
10	0.146	-2.911	-0.932	0.007	5.667	2.047	-11.738	-7.715
12	0.105	-2.725	-0.896	0.004	5.350	1.929	-10.332	-6.565
14	0.080	-2.576	-0.862	0.002	5.047	1.819	-9.218	-5.708
16	0.064	-2.449	-0.829	0.002	4.769	1.718	-8.317	-5.044
18	0.053	-2.338	-0.798	0.001	4.517	1.627	-7.574	-4.514
20	0.044	-2.238	-0.769	0.001	4.289	1.544	-6.953	-4.081

TABLE 1

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FIGURE CAPTIONS

Fig. 1 Deep inelastic scattering on a virtual photon in $e^+e^- \to e^+e^-$ + hadrons. $Q^2(P^2)$ is mass square of the "probe" ("target") photon ($\Lambda^2 << P^2 << Q^2$).

Fig. 2 Forward virtual photon-photon scattering.

Fig. 3 The virtual photon structure function $F_2^\gamma(x,Q^2,P^2)$ to the next-to-leading order (H.O.) in units of $3f\langle e^4\rangle(\alpha/\pi)\ln Q^2/P^2$ for Q^2=30 GeV^2, P^2=1 GeV^2 with (a) $\Lambda_{\overline{MS}}$=500 MeV, (b) $\Lambda_{\overline{MS}}$ =100 MeV (solid lines), together with the real photon structure function $F_2^\gamma(x,Q^2)$ for Q^2=30 GeV^2 (dashed lines) which can be formally reproduced by putting $P^2=\Lambda^2$. We also plot the leading-log (L.O.) results of ref. [11] (solid line for P^2=1 GeV^2, dashed-dotted line for real γ). Here, in these analyses, we have taken f=4.

Fig. 4 (a) $D_\gamma(x)$.

(b) $A_-(x)$.

Fig. 5 The longitudinal virtual photon structure function $F_L^\gamma(x,Q^2,P^2)$ in units of $3f\langle e^4\rangle(\alpha/\pi)$ for Q^2=30 GeV^2, P^2=1 and 4 GeV^2 with $\Lambda_{\overline{MS}}$=100 MeV (solid lines). We have also shown the longitudinal real photon structure function $F_L^\gamma(x,Q^2)$ of the box diagram (dashed line) and that with QCD correction (dashed-dotted lines).

Fig. 6 Comparison of the box contribution with the QCD prediction for the virtual photon structure function $F_2^{\gamma}(x,Q^2,P^2)$ in units of $3f\langle e^4\rangle(\alpha/\pi)\ln Q^2/P^2$ for $Q^2=30\ \mathrm{GeV}^2$, $P^2=1\ \mathrm{GeV}^2$ and $\Lambda_{\overline{MS}}=100\mathrm{MeV}$. The results are shown both in the leading-log order and in the next-to-leading order. The box contribution including the next-to-leading-log term is given by eq.(5.3).

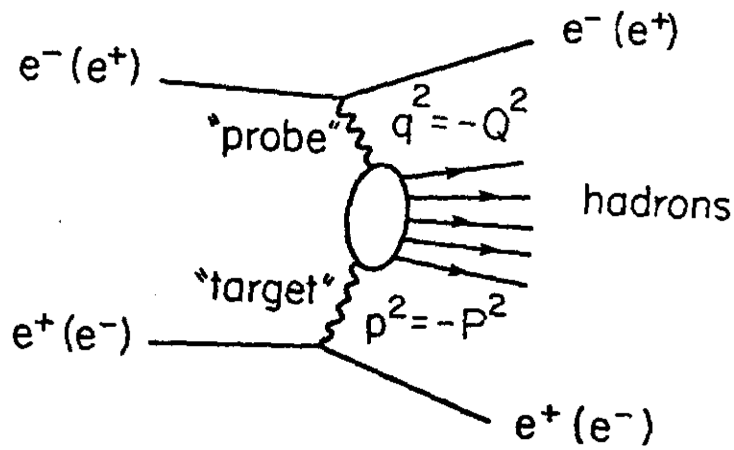


Figure 1

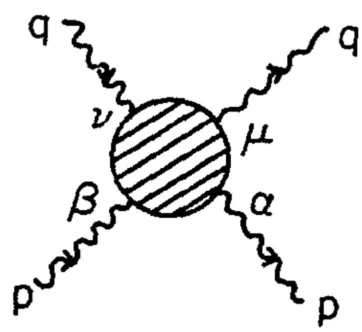


Figure 2

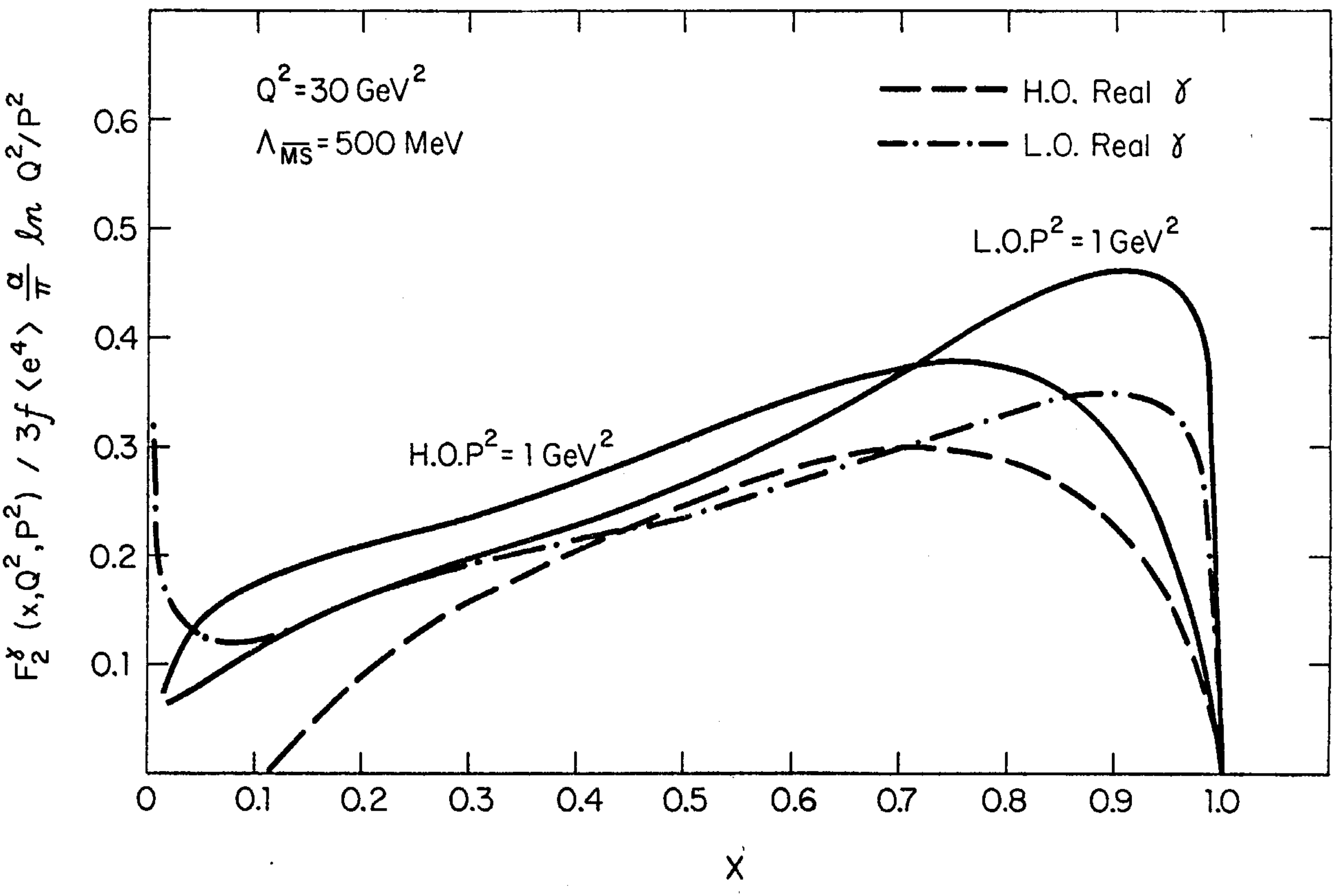
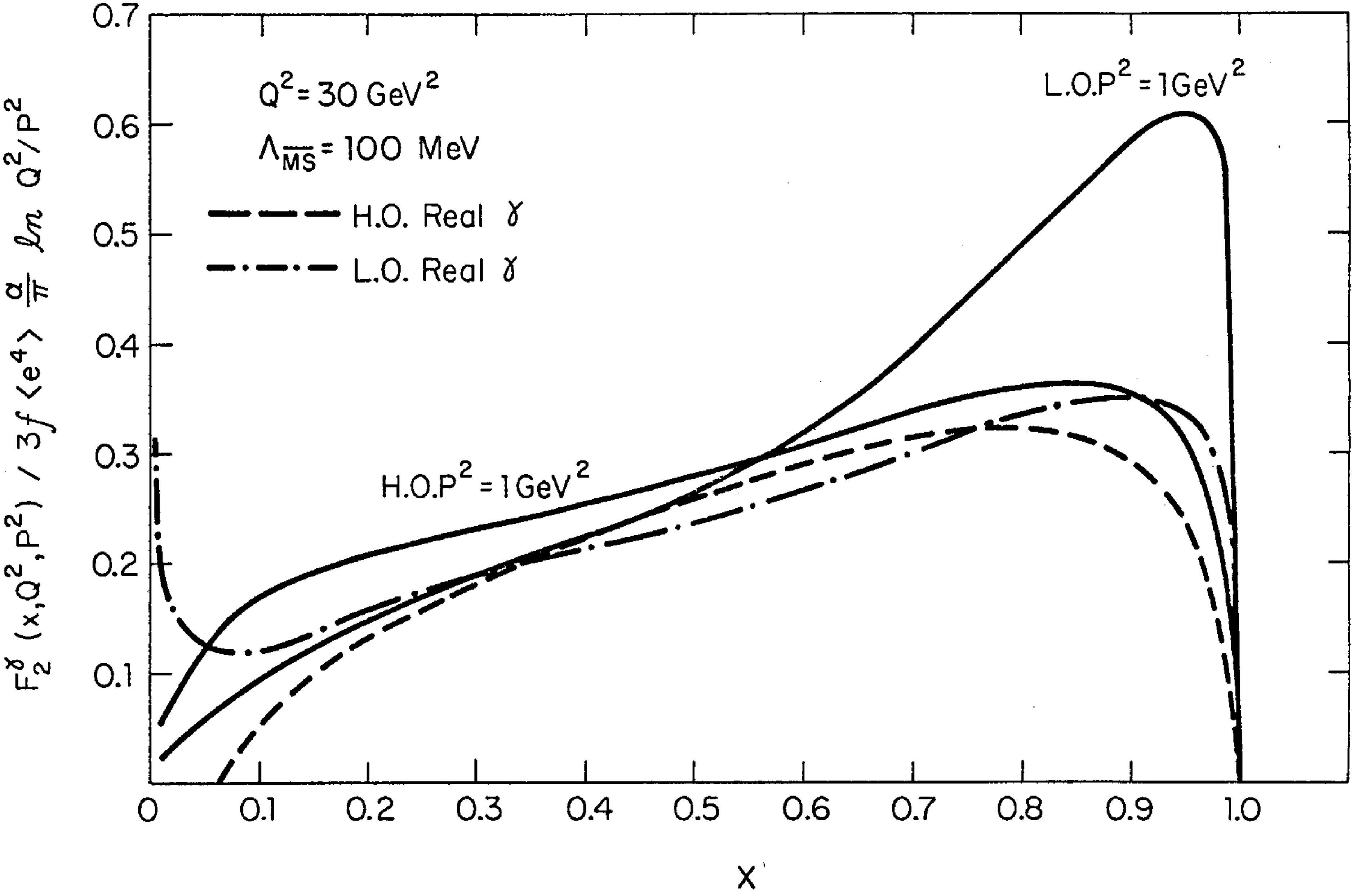

$Q^2 = 30\ GeV^2$
$\Lambda_{\overline{MS}} = 500\ MeV$
H.O. Real γ
L.O. Real γ
L.O. $P^2 = 1\ GeV^2$
H.O. $P^2 = 1\ GeV^2$
$F_2^\gamma(x, Q^2, P^2)\ /\ 3f\langle e^4\rangle\ \frac{\alpha}{\pi}\ \ln\ Q^2/P^2$
0.1
0.2
0.3
0.4
0.5
0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X

Figure 3a



$Q^2 = 30\ GeV^2$
$\Lambda_{\overline{MS}} = 100\ MeV$
H.O. Real γ
L.O. Real γ
L.O.$P^2 = 1GeV^2$
H.O.$P^2 = 1GeV^2$
$F_2^\gamma(x, Q^2, P^2) / 3f \langle e^4 \rangle \frac{\alpha}{\pi} \ln Q^2/P^2$
X

Figure 3b

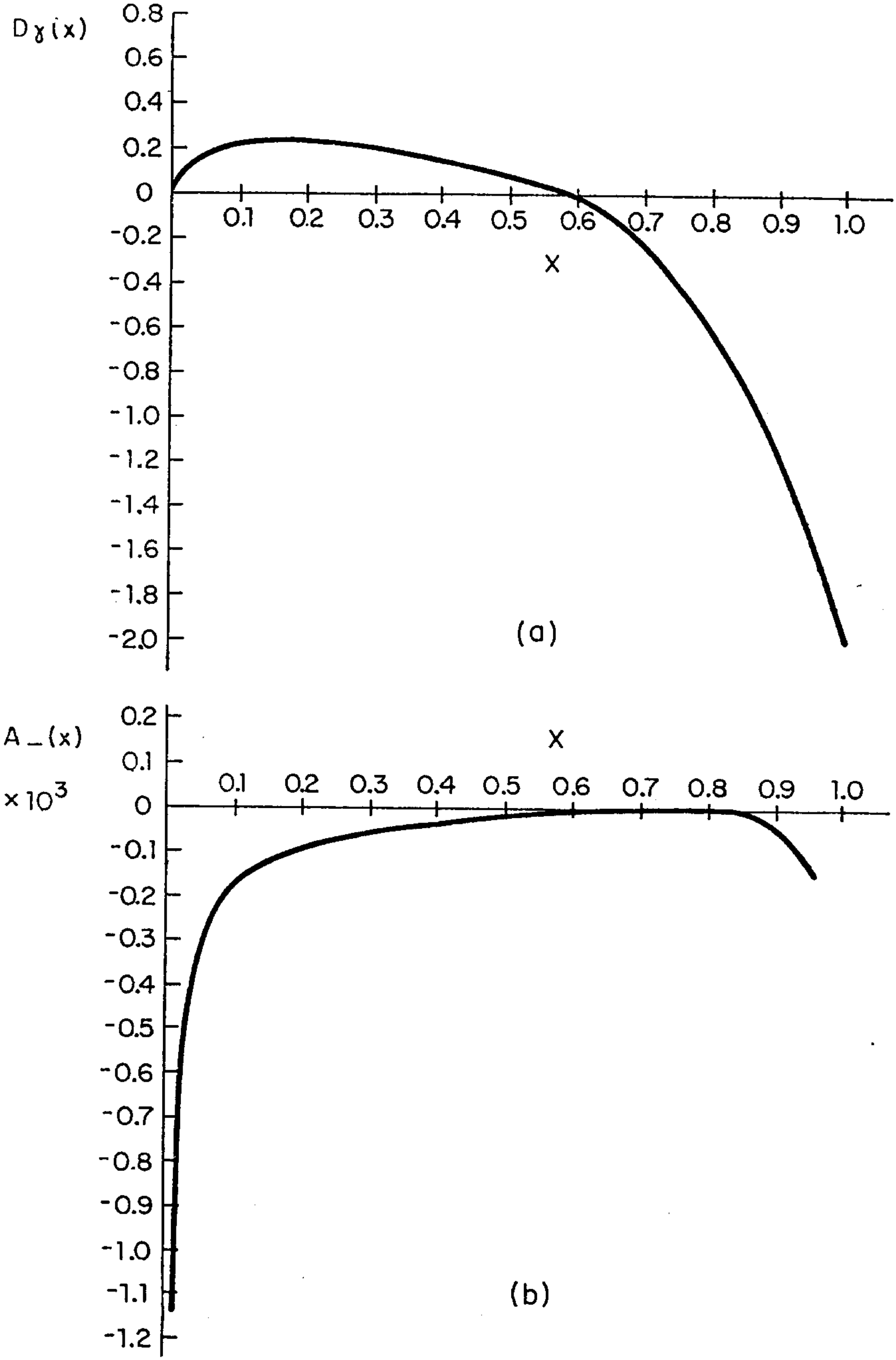

Dγ(x)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
X
(a)
A_(x)
×10³
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
X
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
(b)

Figure 4

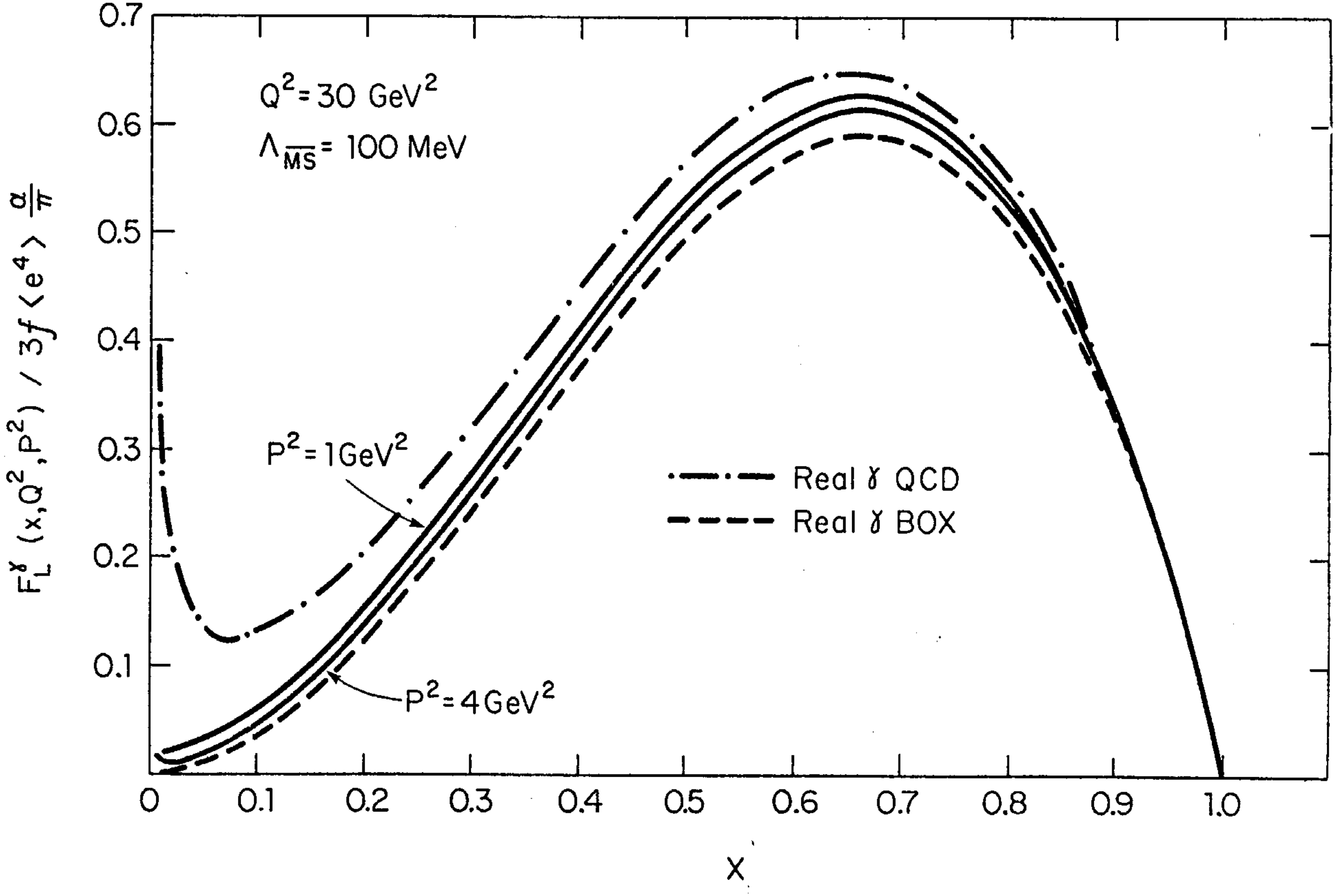

$Q^2 = 30\ GeV^2$
$\Lambda_{\overline{MS}} = 100\ MeV$
$P^2 = 1 GeV^2$
$P^2 = 4 GeV^2$
Real γ QCD
Real γ BOX
$F_L^\gamma\ (x,Q^2,P^2)\ /\ 3f\langle e^4\rangle\ \frac{\alpha}{\pi}$
X
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure 5

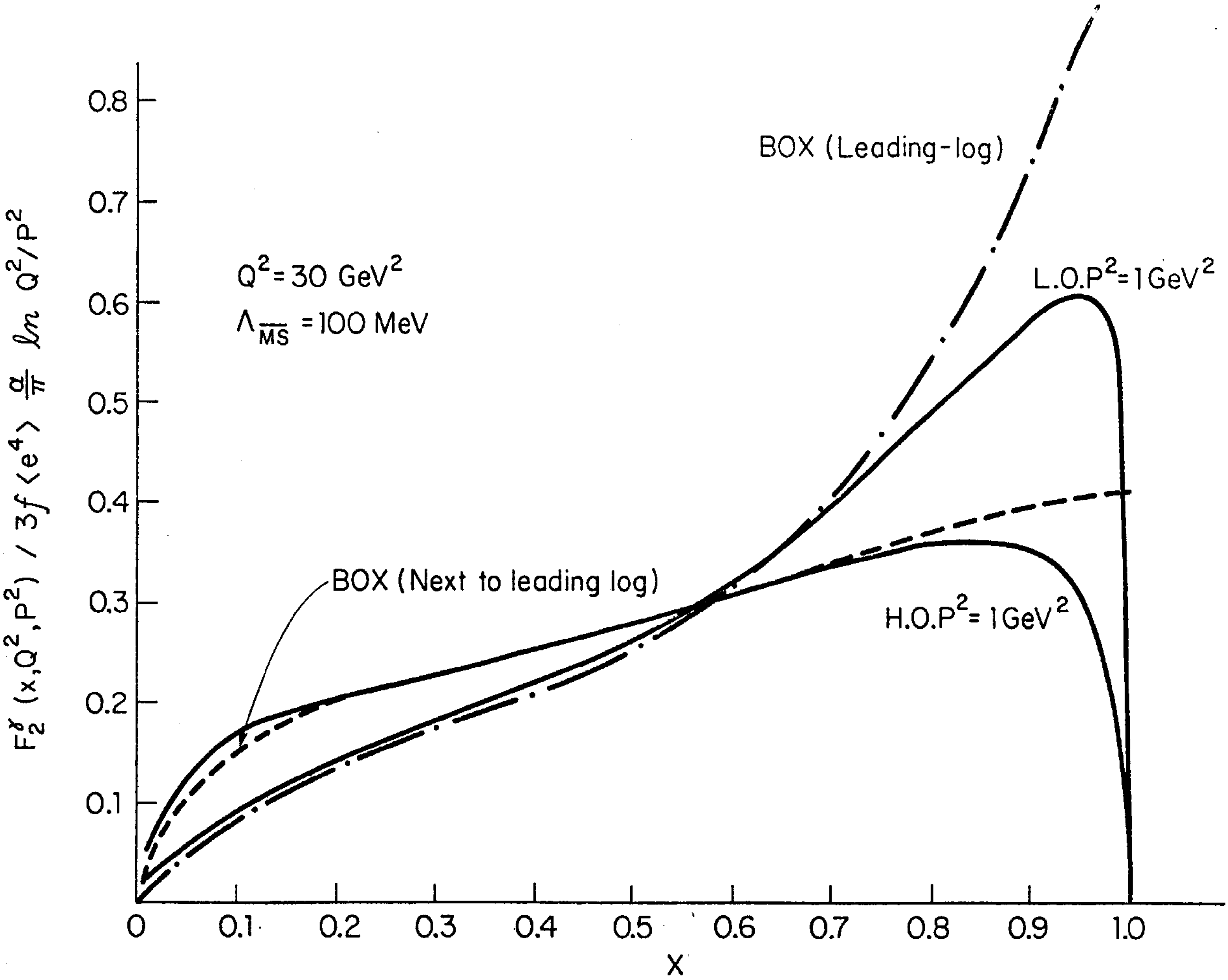

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
$F_2^\gamma(x,Q^2,P^2)\ /\ 3f\langle e^4\rangle\ \frac{\alpha}{\pi}\ \ln\ Q^2/P^2$
$Q^2 = 30\ GeV^2$
$\Lambda_{\overline{MS}} = 100\ MeV$
BOX (Leading-log)
$L.O.P^2 = 1 GeV^2$
BOX (Next to leading log)
$H.O.P^2 = 1 GeV^2$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
X

Figure 6